Calculation and Application of the Density Propagator for a Self-Bound Fermion System. I. Theory*†

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The Martin-Schwinger-Puff theory for a many-fermion system is applied to a calculation of the appropriate density propagator. The real part of this function, the system's polarizability, is a measure for the lowest order change in energy due to a harmonic density ripple enforced on the infinite system. The imaginary part, the linear response function, yields the inelastic cross section for the scattering of a weak external agent. The linear response contains additional information on excited states and, in particular, on collective states, excited by an external field. The formalism necessitates the calculation of the self-energy of a particle and from it the optical model for nucleon scattering can be derived. The relation of the theory to perturbative and other nonperturbative approaches is discussed and a comparison is made with a calculation of the polarization energy of nuclear matter by Koltun and Wilets.

1. INTRODUCTION

A GREAT variety of techniques has been developed in recent years as tools for the study of many-fermion systems.[1] As far as these systems are non-superconducting most of the techniques have been devised to expand primary quantities, such as the ground-state energy, in a perturbation series (partially summed eventually) in terms of a two-body interaction. This common point in method is also reflected in the basic results obtained: There is virtually no difference in the description of, say, the plasmon state of an electron gas and the ensuing screening of the bare interaction between electrons whether the approach is a dielectric formulation,[2] the random phase approximation,[3] Feynman diagram techniques,[4] or a linearization of equations of motion.[5] It is only in finer details that results obtained by different theories start to diverge. In a novel approach to the many-body problem, which is nonperturbative and which resembles cluster expansions commonly used in statistical mechanics, one expresses ground-state properties in terms of Green's functions. In a rigorous theory functions of various orders are mutually connected and in any approximation scheme one has to decide where recursive relations are replaced by a set of self-contained soluble equations.

In the Martin-Schwinger-Puff theory,[6,7] which we shall apply below, stress is laid on the fact that Green's functions satisfy simple boundary conditions provided those functions are defined as grand canonical ensemble averages. Ground-state properties are then defined as properties of the system in the zero-temperature limit.

The theory has been applied successfully to a calculation of the binding energy and density of nuclear matter in its ground state.[7] The underlying approximation amounts to the neglect of correlations between more than two particles, retaining the effect of correlations between two particles in a certain scattering matrix. The way the approximations are made, however, spoils the fundamental symmetry between the particles and has been shown to amount to an average treatment of statistics instead of a rigorous one.[7,8] The satisfactory results obtained for energy and density encourage one in the belief that the approximations are adequate for the conditions that prevail in nuclear matter and that, consequently, one should take full advantage of the easily soluble model that results.

In what follows we present a calculation of the density propagator for a self-bound fermion system. This quantity, which is closely related to the generalized dielectric constant, enables the calculation of several characteristic properties of the system in its ground state which did not receive attention by Puff. We first mention the polarizability of the medium, the importance of which has recently been stressed.[9,10]

A second problem of interest is the existence of anomalous bound states as occurring in superconductors on one hand[11] and of collective states like zero sound on the other hand.[12,13]

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[1] K. Brueckner, Phys. Rev. **100**, 36 (1955) and many other papers; K. Brueckner and S. L. Gammel, *ibid.* **109**, 1038 (1958); J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957); N. M. Hugenholtz, Physica **23**, 481 (1957); N. M. Hugenholtz and L. van Hove, *ibid.* **24**, 363 (1958); R. E. Prange and A. B. Klein, Phys. Rev. **112**, 994 (1958); A. B. Klein, in *Lectures on Field Theory and the Many-Body Problem, Naples, 1960* [Academic Press Inc., New York (to be published)].

[2] P. Nozières and D. Pines, Nuovo Cimento **9**, 470 (1958).

[3] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).

[4] J. Hubbard, Proc. Roy. Soc. (London) **A243**, 336 (1957); D. F. du Bois, Ann. Phys. (N. Y.) **7**, 174 (1959); **8**, 24 (1959).

[5] K. Sawada, Phys. Rev. **106**, 372 (1957); K. Sawada, K. A. Brueckner, N. Fukuda, and R. Brout *ibid.* **108**, 507 (1957).

[6] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

[7] R. D. Puff, Ann. Phys. (N. Y.) **13**, 317 (1961).

[8] D. S. Falk and L. Wilets, Phys. Rev. **124**, 1887 (1961).

[9] D. Koltun and L. Wilets, preceding paper [Phys. Rev. **129**, 880 (1962)].

[10] S. Moszkowski and L. Wilets (to be published).

[11] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).

[12] L. Van Hove, Physica **25**, 849 (1959).

[13] K. Gottfried and L. Pičman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **32**, No. 13 (1960).

We define in Sec. 2 below the density propagator for a system in its ground state, where the latter is defined as the state to which the system approaches in the limit $T\rightarrow 0$. In case such a unique state exists, the relations with the system's polarizability and response to an external field is established in the usual way.

Section 3 contains a compilation without rigorous proofs of statistical Green's functions, their relations and the Martin-Schwinger-Puff (MSP) approximation which enable a calculation of one-body and two-body Green's functions. It is then shown in Sec. 4 how the density propagator can be determined from a two-body Green's function with a special choice of coordinates. Consequent use of the MSP approximation yields a closed expression in terms of functions which have been or may be calculated for a given two-body interaction.

In Sec. 5 we recall the relation between the response function on one hand and the spectral function and various physical quantities on the other. Mention is made in particular of the feasibility of a calculation of the optical potential for scattering of nucleons off nuclear matter.

In the last section finally a comparison is given with the theories of Gottfried and Pičman[13] and of Koltun and Wilets.[9] Results of actual calculations will be presented in a separate paper.

2. THE GROUND-STATE DENSITY PROPAGATOR

We shall be concerned in the following with some ground-state properties of a many-fermion system. The particular Green's function technique which we shall invoke for the determination of those properties is most powerful if first grand ensemble averages over states are taken. We thus consider the grand canonical weighting operator

$$P=\lim_{\Omega\rightarrow\infty} Z^{-1}(i\tau,\mu,\Omega)\exp[-i\tau(H-\mu\mathfrak{N})], \quad (1)$$

with H the Hamiltonian of the system including two-body forces and $\mathfrak{N}$ the particle operator.[14] Ω and μ are the volume and the chemical potential, respectively. Instead of the temperature, $i\tau=(kT)^{-1}$ is used for convenience. Z finally is the grand canonical partition function, which properly normalizes the distribution. Averages of operators X are then defined by means of (1) as

$$\langle X\rangle=\lim_{\Omega\rightarrow\infty} Z^{-1}(i\tau,\mu,\Omega)\,\mathrm{Tr}\{\exp[-i\tau(H-\mu\mathfrak{N})]X\}. \quad (2)$$

The ground state is now defined as the one singled out in the limit $i\tau\rightarrow\infty$. Such a definition is possible provided $H-\mu\mathfrak{N}$ has a unique lower bound, in which case we call $E_0(N_0)/\Omega$ and N_0/Ω the values of $\langle H\rangle/\Omega$ and $\langle\mathfrak{N}\rangle/\Omega$ which dominate for $i\tau\rightarrow\infty$. Equation (2) then reduces to

$$\langle X\rangle_0=z^{-1}\sum_{\zeta_0}\langle N_0E_0(N_0)\zeta_0|X|N_0E_0(N_0)\zeta_0\rangle, \quad (2a)$$

ζ is reserved for all quantum numbers other than N and E: ζ_0 thus accounts for degeneracies in the ground state, the degree of which is denoted by z.

The chemical potential in the limit $i\tau\rightarrow\infty$ can be shown to equal[6,7]

$$\mu=\frac{\partial E_0(N_0)}{\partial N_0}=\frac{E_0(N_0)}{N_0}+\frac{P}{\rho}, \quad (3)$$

with $\rho=N_0/\Omega$ and P the pressure. A system such as nuclear matter has the property of binding itself at a certain density, $\rho_0\neq 0$, in absence of external pressure; this property renders $\mu_0=E_0(N_0)/N_0<0$. The binding forces in the nuclear case are even strong enough to make $\mu_0<-\frac{1}{2}\epsilon_B$, with $-\epsilon_B$ the energy of the lowest bound state of the two-particle system. Both properties considerably simplify the calculation of ground-state properties like energy and density.

The quantity which we shall discuss below is the propagator for density fluctuations in the ground state.[13,6,15] Its definition reads

$$D(\mathbf{q}\omega)=\langle\mathfrak{D}(\mathbf{q}\omega)\rangle_0=-i\lim_{i\tau\rightarrow\infty}\int\int d(t_1-t_2)d(\mathbf{x}_1-\mathbf{x}_2)$$
$$\times e^{i\omega(t_1-t_2)}e^{-i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\langle T\{\delta\rho(1)\delta\rho(2)\}\rangle. \quad (4)$$

T in (4) denotes Wick's time-ordering operator, and $i\equiv\mathbf{x}_it_i$. $\delta\rho(1)$ is the operator describing density fluctuations from its average, $\langle\rho(x_1)\rangle_0\equiv\langle\sum_n\delta(\mathbf{x}_n-\mathbf{x}_1)\rangle_0$ and is given by

$$\delta\rho(1)=e^{iHt_1}\rho(x_1)e^{-iHt_1}-\langle\rho(x_1)\rangle_0. \quad (5)$$

Again the zero-temperature limit will single out the ground state, such that

$$D(\mathbf{q}\omega)=-i\int d(t_1-t_2)d(\mathbf{x}_1-\mathbf{x}_2)\,e^{i\omega(t_1-t_2)}e^{-i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}$$
$$\times z^{-1}\sum_{\zeta_0}\langle N_0E_0(N_0)\zeta_0|T\{\delta\rho(1)\delta\rho(2)\}|$$
$$\times N_0E_0(N_0)\zeta_0\rangle_0. \quad (6)$$

Since the operator $\delta\rho$ does not change the number of particles, one can work in the subspace of N_0 particles. Let us, in that subspace, denote by $|n\zeta_n\rangle$ and ω_{n0}, respectively, the states and excitation energies reached by $\delta\rho$ out of the ground state. It is then an easy matter to expand (6) in terms of the complete set $|n\zeta_n\rangle$ leading to

$$D(\mathbf{q}\omega)=\lim_{\epsilon\rightarrow 0}z^{-1}\sum_{\zeta_0,n,\zeta_n}\left\{\frac{|\langle n\zeta_n|\rho_\mathbf{q}|0\zeta_0\rangle|^2}{\omega-\omega_{n0}+i\epsilon}\right.$$
$$\left.-\frac{|\langle n\zeta_n|\rho_\mathbf{q}^\dagger|0\zeta_0\rangle|^2}{\omega-\omega_{n0}-i\epsilon}\right\}+2\pi i\delta(\omega)z^{-1}$$
$$\times\sum_{\zeta_0}|\langle 0\zeta_0|\rho_\mathbf{q}|0\zeta_0\rangle|^2, \quad (7)$$

[14] Units $\hbar=2m=1$ are used throughout.

[15] L. van Hove, Phys. Rev. 95, 249 (1954).

where instead of $\rho(x)$ we introduced its Fourier component $\rho_{\mathbf{q}}=\sum_n e^{i\mathbf{q}\mathbf{x}_n}$.

The importance of $D(\mathbf{q}\omega)$ as a source of information for nuclear properties is well known. Consider, for instance, the system subject to an interaction through a weakly polarizing density fluctuation of strength a_0:

$$V=V^\dagger=\tfrac{1}{2}a_0(\rho_{\mathbf{q}}+\rho_{\mathbf{q}}^\dagger). \tag{8}$$

The ensuing polarization energy is defined as the second order shift in the ground-state energy. Strictly speaking one has to define the perturbed ground state from the beginning, replacing everywhere the Hamiltonian by $H+V$. This program has been carried out by Koltun and Wilets[9] and a comparison will be deferred to Sec. 6. The energy shift due to V, Eq. (8), is trivally seen to be[16]

$$E^{(2)}\equiv-\tfrac{1}{2}[\alpha(\mathbf{q})+\alpha(-\mathbf{q})]a_0^2=-\tfrac{1}{4}a_0^2z^{-1}\times\sum_{n\neq0;\zeta_n,\zeta_0}{}'\frac{|\langle n\zeta_n|\rho_{\mathbf{q}}|0\zeta_0\rangle|^2+|\langle n\zeta_n|\rho_{\mathbf{q}}^\dagger|0\zeta_0\rangle|^2}{\omega_{n0}}, \tag{9}$$

with $\alpha(\mathbf{q})$ the static polarizability of the medium in its ground state,[16]

$$\alpha(\mathbf{q})=\tfrac{1}{2}z^{-1}\sum_{n\neq0;\zeta_0,\zeta_n}{}'\frac{|\langle n\zeta_n|\rho_{\mathbf{q}}|0\zeta_0\rangle|^2}{\omega_{n0}}. \tag{10}$$

One determines in the same fashion the lowest order change in the $\mathbf{k}$ Fourier component of the particle density, namely,

$$\rho^{(1)}(\mathbf{k})=-a_0\{\alpha(\mathbf{q})+\alpha(-\mathbf{q})\}\delta^{(3)}(\mathbf{k}-\mathbf{q}). \tag{11}$$

It is now an easy matter to connect $\alpha(\mathbf{q})$ with the density propagator, and one has in fact

$$\alpha(\mathbf{q})+\alpha(-\mathbf{q})=-\mathrm{Re}D(\mathbf{q}0)=-D(\mathbf{q}0). \tag{12}$$

Also the absorptive part of D has physical meaning. Denoting by θ the usual step function, one infers from (7) that

$$R(\mathbf{q}\omega)\equiv\pi^{-1}\,\mathrm{Im}D(\mathbf{q}\omega)\theta(\omega)=z^{-1}\sum_{n,\zeta_n,\zeta_0}|\langle n\zeta_n|\rho_{\mathbf{q}}|0\zeta_0\rangle|^2\delta(\omega-\omega_{n0}), \tag{13}$$

which function describes the system's response to the $\mathbf{q}$ Fourier component of any external field. $R(\mathbf{q}\omega)$ is, apart from a trivial factor, the inelastic cross section in Born approximation for scattering of a particle or field by a system to which it imparts momentum $\hbar\mathbf{q}$ and energy $\hbar\omega$. This quantity contains information of paramount importance such as the excitation spectrum and transition probabilities from the ground state induced by an external field, and is measurable provided the conditions for which the Born approximation holds actually prevail. For an exhaustive and lucid exposition of the theory of the linear response function for a many-fermion system, the reader is referred to a recent review article by Glick.[17]

One of the objectives of the study of the response function $R(\mathbf{q}\omega)$ has already been mentioned, namely, the detection of possible collective states.

One notices, incidently, that the existence of a collective density fluctuation bears also on the interpretation of the static polarizability $\alpha(\mathbf{q})$, Eq. (10). In case zero sound exists, it contributes in a collective way to the weighted transition probabilities as appearing in $\alpha(\mathbf{q})$. The intimate relation between $\alpha(\mathbf{q})$ and $D(\mathbf{q}\omega)$ can also be expressed in terms of a dispersion relation, to be found from (7) and (10):

$$\alpha(\mathbf{q})+\alpha(-\mathbf{q})=\frac{1}{2\pi}P\left[\int_0^\infty\frac{\mathrm{Im}D(\mathbf{q}\omega)}{\omega}d\omega-\int_{-\infty}^0\frac{\mathrm{Im}D(\mathbf{q}\omega)}{\omega}d\omega\right]. \tag{14}$$

Since we wish to express the density propagator (4) in terms of Green's functions, we first briefly review definitions and some properties without going into much detail or providing proofs. Extensive discussions can be found in the work of Martin and Schwinger,[6] Puff,[7] and Koltun and Wilets.[9]

3. STATISTICAL GREEN'S FUNCTIONS AND THE MARTIN-SCHWINGER-PUFF APPROXIMATION

The n-body Green's function in the limit $i\tau\to\infty$ is defined as

$$\begin{aligned}G_n(1\cdots n,1'\cdots n')&=\lim_{i\tau\to\infty}(-i)^n\langle T\{\psi(1)\cdots\psi(n)\psi^\dagger(n')\cdots\psi(1')\}\rangle\\&=\lim_{i\tau\to\infty}\lim_{\Omega\to\infty}Z^{-1}(i\tau,\mu_1\Omega)(-i)^n\,\mathrm{Tr}[\exp(-i\tau(H-\mu\mathfrak{N}))\times T\{\psi(1)\cdots\psi(n)\psi^\dagger(n')\cdots\psi^\dagger(1')\}];\end{aligned} \tag{15}$$

A boundary condition for all G_n follows from the cyclic property of the trace operation in (15), viz.,

$$G_n(\cdots,r_i\tau,\cdots)=-G_n(\cdots r_i0,\cdots);\quad 0\leqslant t_j\leqslant\tau. \tag{16}$$

The simplicity of this boundary condition is a consequence of the use of a grand canonical averaging procedure and is useful enough to justify the study of the ground state as the $i\tau\to\infty$ limit of a finite temperature situation.

One derives from the equation of motion for a field operator, $\dot\psi=i[H,\psi]_-$, a set of coupled equations, which can be given in either differential or integral form. We meniton as particular cases

$$\left(i\frac{\partial}{\partial t_1}+\Delta_1+\mu\right)G_1(11')+i\langle 12|v|34\rangle G_2(34;1'2^+)=\delta^4(11'). \tag{17}$$

[16] The sums in Eqs. (9) and (10) have, in the limit $\Omega\to\infty$, to be looked upon as principal-value integrals.

[17] A. J. Glick, Ann. Phys. (N. Y.) **17**, 61 (1961).

$\langle 12|v|34\rangle$ in (17) stands for a nonlocal potential

$$\langle 12|v|34\rangle=\delta(t_1-t_2)\delta(t_2-t_3)\delta(t_3-t_4)\delta(\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4))\times\langle|\mathbf{x}_1-\mathbf{x}_2||v||\mathbf{x}_3-\mathbf{x}_4|\rangle; \quad (18)$$

t^+ indicates a time infinitesimally larger t. In (17) and henceforth, integration over repeated coordinates is implied. Internal coordinates will, for reasons of simplicity, not be denoted, but their explicit inclusion can always be made in a trivial way.

Apart from $G_1(11')$ as defined in (17), one introduces a convenient, auxiliary quantity $G_1^0(11')$ which is the solution of (17) with v equal to 0.

Corresponding to (17) one has an equation for G_2 which, in integral form, may be written as[7]

$$G_2(12;1'2')=G_1(11')G_1(22')-G(12')G(21')+\tfrac{1}{2}i[G_1^0(13)G_1(24)+G_1(13)G_1^0(24)]\times\langle 34|v|56\rangle G_2(56;1'2')+C. \quad (19)$$

In C are contained Green's functions or correlations of order three which, in turn, are coupled to Green's functions of still higher orders. In any attempt to calculate Green's functions, one interrupts the coupled set of equations of which (17) and (19) are examples. The MSP approximation, for instance, is defined by a neglect of C and a replacement of G_1 inside the bracket in (19) by G_1^0. The first approximation amounts to a neglect of three-particle correlations while for the second approximation there is no reason other than a considerable simplification of the calculations. Together, the approximations amount to a neglect of certain four-particle correlations in a way that spoils the basic symmetry between the particles and which thus indicates an inexact treatment of statistics. Once the MSP approximation is made, one can proceed to determine G_1 and G_2. One first proves that a "wave matrix" Ω, defined by

$$G_2(12;1'2')=\langle 12|\Omega|34\rangle G_1(31')G_1(42')\rangle, \quad (20)$$

satisfies the following integral equation:

$$\langle 12|\Omega|1'2'\rangle=\delta(11')\delta(22')-\delta(12')\delta(21')+iG_1^0(13)G_1^0(24)\langle 34|v|56\rangle\langle 56|\Omega|1'2'\rangle. \quad (21)$$

Next one introduces a scattering matrix T by

$$\langle 34|T|1'2'\rangle=\langle 34|v|56\rangle\langle 56|\Omega|1'2'\rangle, \quad (22)$$

which by use of (20) is seen to be a solution of

$$\langle 12|T|1'2'\rangle=\langle 12|v|1'2'\rangle-\langle 12|v|2'1'\rangle+i\langle 12|v|34\rangle G_1^0(35)G_1^0(46)\langle 56|T|1'2'\rangle. \quad (23)$$

On substituting (23) into (19), there results, after the described approximations,

$$G_2(12;1'2')\approx G_1(11')G_1(22')-G_1(12')G_1(21')+iG_1^0(13)G_1^0(24)\langle 34|T|56\rangle G(51')G(62'). \quad (24)$$

In a similar fashion one extracts from (17), (20), and (22) a differential equation for G_1:

$$(i\partial/\partial t_1+\Delta_1+\mu)G_1(11')+i\langle 12|T|34\rangle G_1(31')G_1(42')=\delta(11'). \quad (25)$$

Space and time translational invariance favors the use of Fourier transforms over the coordinate-time representation of the Green's functions. One shows, for instance, from (16) and (25) with $V=T=0$, that the Fourier transform of $G_1^0(11')$ equals

$$G_1^0(\mathbf{k}\omega)=\int d(\mathbf{x}_1-\mathbf{x}_1')d(t_1-t_1')\times e^{i\omega(t_1-t_1')}e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_1')}G_0(11')=\lim_{\epsilon\to 0}\left(\frac{\theta(\mathbf{k}^2-\mu)}{\omega-\mathbf{k}^2+\mu+i\epsilon}+\frac{\theta(-\mathbf{k}^2+\mu)}{\omega-\mathbf{k}^2+\mu-i\epsilon}\right). \quad (26)$$

Actually for $i\tau\to\infty$, $\mu\to\mu_0<0$ only the forward propagating part of $G_1^0(\mathbf{k}\omega)$ in (26) survives. Thus

$$G_1^0(\mathbf{k}\omega)=\lim_{\epsilon\to 0}(\omega-\mathbf{k}^2+\mu+i\epsilon)^{-1}. \quad (27)$$

One sees here a difference from the Feynman propagator G_1^0 in a perturbative treatment, which is of the form (26) with $\mu>0$. This difference has far-reaching computational consequences.

The next quantities of interest are the Ω and T matrices, which for an instantaneous interaction V, depend on one frequency only. We denote relative momenta by $\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$, $\mathbf{k}'=\frac{1}{2}(\mathbf{k}_1'-\mathbf{k}_2')$; and center-of-mass momenta by $\mathbf{K}=\mathbf{K}'=\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_1'+\mathbf{k}_2'$, which momenta are conjugate to $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{X}$, and $\mathbf{X}'$. In terms of[18]

$$\langle\mathbf{k}|v|\mathbf{k}'\rangle=\int e^{-i\mathbf{k}\cdot\mathbf{x}}\langle\mathbf{x}|v|\mathbf{x}'\rangle e^{i\mathbf{k}'\cdot\mathbf{x}'}d\mathbf{x}d\mathbf{x}', \quad (28)$$

we then find for the Fourier transform of Ω, Eq. (21)

$$\begin{aligned}\langle\mathbf{k}|\Omega_{\mathbf{K}}{}^+(\omega)|\mathbf{k}'\rangle&=\int d\mathbf{X}d\mathbf{x}d\mathbf{x}'dt_1\exp[i\mathbf{K}\cdot(\mathbf{X}-\mathbf{X}')]\exp(-i\mathbf{k}\cdot\mathbf{x})\times\exp(i\mathbf{k}'\cdot\mathbf{x}')\exp[i\omega(t_1-t_1')]\langle 12|\Omega^+|1'2'\rangle\\&=(2\pi)^3[\delta^3(\mathbf{k}-\mathbf{k}')-\delta^3(\mathbf{k}+\mathbf{k}')]+\lim_{\epsilon\to 0}(\omega-\tfrac{1}{2}\mathbf{K}^2-2\mathbf{k}^2+2\mu+i\epsilon)^{-1}\int\langle\mathbf{k}|v|\mathbf{k}''\rangle\frac{d\mathbf{k}''}{(2\pi)^3}\times\langle\mathbf{k}''|\Omega_{\mathbf{K}}{}^+(\omega)|\mathbf{k}'\rangle \quad (29)\end{aligned}$$

$$\langle\mathbf{k}|T_{\mathbf{K}}{}^+(\omega)|\mathbf{k}'\rangle\equiv\int\langle\mathbf{k}|v|\mathbf{k}''\rangle\frac{d\mathbf{k}''}{(2\pi)^3}\langle\mathbf{k}''|\Omega_{\mathbf{K}}{}^+(\omega)|\mathbf{k}'\rangle$$

[18] The Fourier transforms of v, Ω, and T are defined here differently from Puff, in that they possess an additional factor $(2\pi)^3$.

in turn is the Fourier transform of (23) and satisfies the following integral equation:

$$\langle \mathbf{k}|T_{\mathbf{K}}^{+}(\omega)|\mathbf{k}'\rangle = \langle \mathbf{k}|v|\mathbf{k}'\rangle - \langle \mathbf{k}|v|-\mathbf{k}'\rangle + \int \frac{d\mathbf{k}''}{(2\pi)^3} \frac{\langle \mathbf{k}|v|\mathbf{k}''\rangle\langle \mathbf{k}''|T_{\mathbf{K}}^{+}(\omega)|\mathbf{k}'\rangle}{\omega - \frac{1}{2}\mathbf{K}^2 - 2\mathbf{k}''^2 + 2\mu + i\epsilon}. \quad (30)$$

The analytical behavior of $T(z)$, the scattering matrix "off the energy shell" has first been studied by Watson.[19] He proved quite generally that the T matrix is an analytical, bounded function in the complex z plane except for isolated poles and a branch line on the real axis corresponding to bound states and scattering states of the two-body system in the absence of the medium. It is clear from (30) that for $\omega < \omega_{\mathbf{K}} = \frac{1}{2}\mathbf{K}^2 - 2\mu$ is continuous across the real axis. For $\omega > \omega_{\mathbf{K}}$, on the other hand, the peculiar form (27) of G_1^0 causes the appearance of T^+ with "outgoing" waves only. On integrating over ω the device prescribes passing above the singularities of the T matrix.

We shall also need $G_1(\mathbf{k}\omega)$, the Fourier transform of the one-body Green's function. From (25) one derives the equation it satisfies, viz.,

$$G_1(\mathbf{k}_1\omega_1) = \Big[\omega_1 - \mathbf{k}_1^2 + \mu - i \lim_{\theta\to 0^+} \int \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{d\omega_2}{2\pi} e^{i\omega_2\theta} \times \langle \mathbf{k}|T_{\mathbf{K}}^{+}(\omega_1+\omega_2)|\mathbf{k}'\rangle G_1(k_2\omega_2) \Big]^{-1}. \quad (31)$$

It is convenient to use the spectral decomposition of $G_1(\mathbf{k}\omega)$ by use of $A(\mathbf{k}\omega)$, the latter being defined by

$$G_1(\mathbf{k}\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \Big[\frac{A(\mathbf{k}\omega')\theta(\omega')}{\omega - \omega' + i\epsilon} + \frac{A(\mathbf{k}\omega')\theta(-\omega')}{\omega - \omega' - i\epsilon} \Big], \quad (32)$$

and satisfying

$$\int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} A(\mathbf{k}_1\omega_1) = 1. \quad (33)$$

The spectral function is connected with the discontinuity across the real axis of $\hat{G}(\omega)$, the analytical continuation of $G(\omega)$. In fact,

$$A(\mathbf{k}_1\omega_1) = \lim_{\epsilon\to 0} i[\hat{G}(\mathbf{k}_1\,\omega_1 + i\epsilon) - \hat{G}(\mathbf{k}_1\,\omega_1 - i\epsilon)], \quad (34)$$

Returning to (31) one sees that the analytical properties of T enable one to close the ω_2 contour in the upper half plane. After substitution of (32) into (31), there results

$$\hat{G}_1(\mathbf{k}_1\omega_1) = [\omega_1 - \mathbf{k}_1^2 + \mu - \mathcal{U}(\mathbf{k}_1\omega_1)]^{-1}, \quad (35)$$

where $\hat{G}$ has been expressed in terms of the self-energy $\mathcal{U}$ of the particle:

$$\mathcal{U}(\mathbf{k}_1\omega_1) = \int \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{d\omega_2}{2\pi} A(\mathbf{k}_2\omega_2)\theta(-\omega_2) \times \langle \mathbf{k}|T_{\mathbf{K}}(\omega_1+\omega_2)|\mathbf{k}\rangle. \quad (36)$$

Equations (34), (35), and (36) show that for $\omega_1 < 0$, $\mathcal{U}(\mathbf{k}_1\omega_1)$ is also continuous across the real axis and determines the negative-frequency portion of A, viz.,

$$A(\mathbf{k}_1\omega_1)\theta(-\omega_1) = 2\pi\rho(\mathbf{k}_1)\delta(\omega - \omega(\mathbf{k}_1)). \quad (37)$$

$\rho(\mathbf{k}_1)$ in (37), the momentum distribution of a particle, equals

$$\rho(\mathbf{k}_1) = \{1 - [\partial\mathcal{U}(\mathbf{k}_1\omega_1)/\partial\omega_1]_{\omega_1=\omega_1(\mathbf{k}_1)}\}^{-1}, \quad (38)$$

with $\omega(\mathbf{k}_1)$ the negative definite solution of

$$\omega(\mathbf{k}_1) = \mathbf{k}_1^2 - \mu + \mathcal{U}(\mathbf{k}_1\omega_1)\theta(-\omega_1). \quad (39)$$

The particles described by the dispersion relation (39) are thus undamped quasi-particles, owing to the reality of $\mathcal{U}(\mathbf{k}_1\omega_1)\theta(-\omega_1)$.

For a calculation of the ground-state energy and density, knowledge of $G(\mathbf{k}\omega)$ for negative frequencies only is required.[6,7] The simultaneous solution of (31), (36)–(39) then determines the required quantities. However, in what follows we shall also need $G(\mathbf{k}\omega)$ for positive frequencies. One notices that it is possible to substitute (37) into (36) in order to obtain *for all* ω

$$\mathcal{U}(\mathbf{k}_1\omega_1) = + \int \frac{d\mathbf{k}_2}{(2\pi)^3} \rho(\mathbf{k}_2)\langle \mathbf{k}|T_{\mathbf{K}}^{+}(\omega_1 + \omega(\mathbf{k}_2))|\mathbf{k}\rangle. \quad (40)$$

Besides $\mathcal{U}(\mathbf{k}\omega)\theta(-\omega)$ and the quantity $\mathcal{U}(\mathbf{k}\,\omega(\mathbf{k}))\times\theta[-\omega(\mathbf{k})]$ being the real effective potential experienced by a particle in the Fermi sea, we shall have to calculate $\mathcal{U}(\mathbf{k}\omega)$ for $\omega > 0$. Whereas $\mathrm{Re}\mathcal{U}(\mathbf{k}\omega)$ remains continuous, $\mathrm{Im}\mathcal{U}(\mathbf{k}\omega)$ suffers a discontinuity due to the one in T^+. Remarks as to the physical interpretation of $\mathcal{U}(\mathbf{k}\omega)\theta(\omega)$ are deferred to Sec. 5.

In the following section we shall complete a formal derivation of the density propagator in terms of Green's function.

4. CALCULATION OF THE DENSITY PROPAGATOR

We have shown in Sec. 2 that the density propagator in the form of Eq. (6) from which its physical content is most apparent, is the limit $i\tau \to \infty$ of an ensemble average. It is now an easy matter to express $D(\mathbf{q}\omega)$, Eq. (4), in terms of Green's functions defined by (15) in the same limit. Indeed one has for the autocorrelation function occurring in (4)

$$\langle T\{\delta\rho(1)\delta\rho(2)\}\rangle_0 = -G_2(12; 1^+2^+) - \langle\rho(\mathbf{x}_1)\rangle_0\langle\rho(\mathbf{x}_2)\rangle_0. \quad (41)$$

The average densities $\langle\rho(\mathbf{x})\rangle_0$ in turn are also readily identified by means of (15) to be

$$\langle\rho(\mathbf{x}_1)\rangle = -iG(11^+). \quad (42)$$

[19] K. M. Watson, Phys. Rev. **103**, 489 (1956).

We can thus substitute (41) and (42) into (4) and apply (29) in the MSP approximation to show that

$$D(\mathbf{q}\omega)=i\int\int d(\mathbf{x}_1-\mathbf{x}_2)d(\mathbf{t}_1-\mathbf{t}_2)e^{i\omega(t_1-t_2)}e^{-i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}$$
$$\times[G_1^0(13)G_1^0(24)\langle 34|T|56\rangle G_1(51^+)G(62^+)$$
$$-G(12^+)G(21^+)]$$
$$\equiv D_1(\mathbf{q}\omega)+D_2(\mathbf{q}\omega). \tag{43}$$

D_1 and D_2 defined by the two parts in the bracketed expression in (42) will be calculated separately. The first one is readily written as

$$D_1(\mathbf{q}\omega)=-\lim_{\theta_1\to 0^+,\theta_2\to 0^+}\int e^{i\omega_1\theta_1}e^{i\omega_2\theta_2}\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}$$
$$\times G_1^0(\mathbf{k}_1+\mathbf{q}\,\omega_1+\omega)G_1^0(\mathbf{k}_2-\mathbf{q}\,\omega_2-\omega)$$
$$\times\langle\mathbf{k}+\mathbf{q}|T_K(\omega_1+\omega_2)|\mathbf{k}\rangle G_1(\mathbf{k}_1\omega_1)G_1(\mathbf{k}_2\omega_2). \tag{44}$$

On substituting (32) for both G_1 factors one then obtains for (44) by use of (37)

$$D_1(\mathbf{q}\omega)=\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_1)\rho(\mathbf{k}_2)G_1^0(\mathbf{k}_1+\mathbf{q}\,\omega(\mathbf{k}_1)+\omega)$$
$$\times G_1^0(\mathbf{k}_2-\mathbf{q}\,\omega(\mathbf{k}_2)-\omega)$$
$$\times\langle\mathbf{k}+\mathbf{q}|T_K\{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)\}|\mathbf{k}\rangle. \tag{45}$$

Since $\omega(\mathbf{k}_1)$ and $\omega(\mathbf{k}_2)$ are both negative, T for the relevant argument is real. $D_1(\mathbf{q}0)$ and the corresponding part of the polarizability $\alpha_1(\mathbf{q})+\alpha_1(-\mathbf{q})$ are real too, as they should be.

$D_2(\mathbf{q}\omega)$, Eq. (43), is defined as $(-i)$ times the Fourier transform of $G(12^+)G(21^+)$. The result

$$D_2(\mathbf{q}\omega)=-i\lim_{\theta_1\to 0^+}\int\frac{d\omega_1}{2\pi}\frac{d\mathbf{k}_1}{(2\pi)^3}e^{i\omega_1\theta_1}$$
$$\times G_1(\mathbf{k}_1\omega_1)G_1(\mathbf{k}_1+\mathbf{q}\,\omega_1+\omega)$$
$$=\int_0^\infty\frac{d\omega_1}{2\pi}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\rho(\mathbf{k}_1)\left[\frac{A(\mathbf{k}_1+\mathbf{q}\,\omega_1)}{\omega(\mathbf{k}_1)-\omega_1+\omega+i\epsilon}\right.$$
$$\left.+\frac{A(\mathbf{k}_1-\mathbf{q}\,\omega_1)}{\omega(\mathbf{k}_1)-\omega_1-\omega+i\epsilon}\right] \tag{46}$$

is easily obtained by substituting (32) into (46).

We are now in a position to summarize the results for, on one hand the lowest order energy shift $E^{(2)}$ and $\rho^{(1)}$, the density change due to the polarization perturbation (8), and the linear response function $R(\mathbf{q}\omega)$ on the other hand. From (9)–(13) we have

$$E^{(2)}/N=\tfrac{1}{2}a_0^2\rho^{-1}D(\mathbf{q}0),$$
$$\rho^{(1)}=a_0D(\mathbf{q}0),$$
$$R(\mathbf{q}\omega)=-\pi^{-1}\,\mathrm{Im}D(\mathbf{q}\omega)\theta(\omega), \tag{47}$$

while $D(\mathbf{q}\omega)$ in the MPS approximation appears in Eqs. (45) and (46) as

$$D(\mathbf{q}\omega)=\int\frac{d\mathbf{k}_1}{(2\pi)^3}\rho(\mathbf{k}_1)\left[\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)\right.$$
$$\times G_1^0(\mathbf{k}_1+\mathbf{q}\,\omega(\mathbf{k}_1)+\omega)G_1^0(\mathbf{k}_2-\mathbf{q}\,\omega(\mathbf{k}_2)-\omega)$$
$$\times\langle\mathbf{k}+\mathbf{q}|T_K\{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)\}|\mathbf{k}\rangle$$
$$+\int_0^\infty\frac{d\omega_1}{2\pi}\left(\frac{A(\mathbf{k}_1+\mathbf{q}\,\omega_1)}{\omega(\mathbf{k}_1)-\omega_1+\omega+i\epsilon}\right.$$
$$\left.\left.+\frac{A(\mathbf{k}_1-\mathbf{q}\,\omega_1)}{\omega(\mathbf{k}_1)-\omega_1-\omega+i\epsilon}\right)\right]. \tag{48}$$

G_0 is defined by (27), while knowledge of $\rho(\mathbf{k})$ and $\omega(\mathbf{k})$ requires both a general solution of Eq. (30) and of the set (36)–(39). ρ and ω have been given by a convenient polynomial fit to a numerical solution in powers of k/k_F for an interaction used in Puff's work.[8] We have those solutions in mind wherever $\rho(\mathbf{k})$ and $\omega(\mathbf{k})$ appear in the following.

The only new feature is the appearance in (48) of the positive frequency portion of the spectral function A. We stress, however, the fact that $A(\mathbf{k}\omega)\theta(\omega)$ is, in principle, not more difficult to calculate than $A(\mathbf{k}\omega)\theta(-\omega)$ and its determination is from (31) and (34) seen to amount to a calculation of the discontinuity of ImG or T^+ across the real axis.

Before closing this section we wish to mention an approximative calculation of (46) based on the following identity, which results from (34) and (35)[6]:

$$A(\mathbf{k}\omega)=\Gamma(\mathbf{k}\omega)/\{[\omega-\epsilon(\mathbf{k}\omega)]^2+\tfrac{1}{4}\Gamma^2(\mathbf{k}\omega)\},$$
$$\epsilon(\mathbf{k}\omega)=\mathbf{k}^2-\mu+\mathrm{Re}\mathcal{U}(\mathbf{k}\omega),$$
$$\Gamma(\mathbf{k}\omega)=2\mathrm{Im}\mathcal{U}(\mathbf{k}\,\omega-i\epsilon). \tag{49}$$

In case $[\partial\Gamma(\mathbf{k},\omega)/\partial\omega]_{\omega(\mathbf{k})}\ll 1-[\partial\epsilon(\mathbf{k},\omega)/\partial\omega]_{\omega(\mathbf{k})}$, one may approximate $A(\mathbf{k}\omega)$ by

$$A(\mathbf{k}\omega)\approx 2\pi\delta(\omega-\epsilon(\mathbf{k},\omega))=2\pi\rho(\mathbf{k})\delta(\omega-\omega(\mathbf{k})), \tag{50}$$

$\omega(\mathbf{k})$ being the root of $\omega=\epsilon(\mathbf{k}\omega)$ and

$$\rho(\mathbf{k})=\{1-[\partial\epsilon(\mathbf{k},\omega)/\partial\omega]_{\omega(\mathbf{k})}\}^{-1}.$$

Substitution of (50) into (46) then yields

$$D_2(\mathbf{q}\omega)\approx\int\frac{d\mathbf{k}_1}{(2\pi)}\rho_<(\mathbf{k}_1)\left[\frac{\rho_>(\mathbf{k}_1+\mathbf{q})}{\omega(\mathbf{k}_1)-\omega(\mathbf{k}_1+\mathbf{q})+\omega+i\epsilon}\right.$$
$$\left.+\frac{\rho_>(\mathbf{k}_1-\mathbf{q})}{\omega(\mathbf{k}_1)-\omega(\mathbf{k}_1-\mathbf{q})-\omega+i\epsilon}\right], \tag{46a}$$

where $<$ and $>$ signs are intended to recall that the densities ρ stem from states with $\omega<0$ and $\omega>0$. If, in particular, $A(\mathbf{k}\omega)$ is replaced by $A^0(\mathbf{k}\omega)=2\pi\delta(\omega-\mathbf{k}^2+\mu)$, which is the spectral function corresponding to $G_1^0(\mathbf{k}\omega)$,

Eq. (27), D_2 attains the form

$$D_2(\mathbf{q}\omega)\approx\int\frac{d\mathbf{k}_1}{(2\pi)^3}\rho_<(\mathbf{k}_1)\Big[\frac{1}{\omega(\mathbf{k}_1)-(\mathbf{k}_1+\mathbf{q})^2+\omega+\mu+i\epsilon}+\frac{1}{\omega(\mathbf{k}_1)-(\mathbf{k}_1-\mathbf{q})^2-\omega+\mu+i\epsilon}\Big]. \quad (46b)$$

Both (46a) and (46b) are typical Hartree-Fock approximations to $D(\mathbf{q}\omega)$ in that the explicit interaction part D_1 does not occur. The expressions are still better than the simple Kramers-Heisenberg form of the polarizability (see for instance references 2 and 3) in that at least one particle state appearing in (46a,b) is treated self-consistently and has a nontrivial energy-momentum relation.

5. QUANTITIES CORRELATED WITH THE DENSITY PROPAGATOR AND THE SPECTRAL FUNCTION

We wish to recall in this section the relation between various frequency moments of the ground-state response function $R(\mathbf{q}\omega)$, Eq. (13), and other physical quantities. We first consider the first moment being the total oscillator strength or, apart from a trivial factor, the total inelastic cross section weighted by the energy transfer. It is well known that for velocity-independent forces this quantity is independent of particle dynamics.[17] In fact

$$\int_{-\infty}^{\infty}R(\mathbf{q}\omega)\omega d\omega=\langle[\rho_{\mathbf{q}}^{\dagger},[H,\rho_{\mathbf{q}}]_-]_-\rangle_0=\rho\mathbf{q}^2. \quad (51)$$

Equation (51) yields no new information but provides instead an additional test as to the accuracy of the approximation scheme.

The inverse first moment is from (14) and (13) seen to be related to the system's static polarizability. Since calculation of the latter has been one of the aims of this paper, there is no need for further discussion here.

Finally the unweighted frequency integral of $R(\mathbf{q}\omega)$ is related to the pair-correlation function[7,17] whose Fourier transform equals (internal variables ζ restored)

$$\begin{aligned}\rho P(\mathbf{q})&\equiv z^{-1}\textstyle\sum_{\zeta_0}\rho P_{\zeta_0}(\mathbf{q})\\&=z^{-1}\textstyle\sum_{\zeta_0}\langle 0\zeta_0|\sum_{i\neq j}^{N}e^{i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)}|0\zeta_0\rangle\\&=\int_{-\infty}^{\infty}R(\mathbf{q}\omega)d\omega-\rho. \quad (52)\end{aligned}$$

$P(\mathbf{q})$ can thus be calculated in the MSP approximation by use of (14), (45), and (47). After some algebra and applying (29) and (33), there results

$$\rho P_{\zeta_0}(\mathbf{q})=\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\times\langle\mathbf{k}+\mathbf{q}|\Omega_K^{\zeta_0}\{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)\}|\mathbf{k}\rangle\rho(\mathbf{k}_1)\rho(\mathbf{k}_2)-(2\pi)^3\delta^3(\mathbf{q})\rho^2. \quad (53)$$

This is essentially the result which was obtained by Puff in a different way.[7]

The quantities mentioned above are related to the two-body Green's function for a special coordinate-time arrangement. It may be mentioned that there is also additional information in the one-body Green's function. The momentum distribution for instance has already been determined from the negative frequency portion of the spectral function $A(\mathbf{k}\omega)$,[7,8] Eqs. (37)–(39). Unfortunately, one is unable, within the MSP approximation, to study the interesting high momentum components of $\rho(\mathbf{k})=\int_{-\infty}^{0}A(\mathbf{k}\omega)d\omega/2\pi$ since the theory predicts for $\omega<0$ a fixed relationship $\omega=\omega(\mathbf{k})$: the upper limit $\omega=0$ also determines a cutoff momentum $\mathbf{k}_f$.

We now return to the calculation of $D(\mathbf{q}\omega)$ where we came across the particle's self-energy $\mathcal{U}(\mathbf{k}\omega)\theta(\omega)$ as appearing in the one-body Green's function $\hat{G}(\mathbf{k}\omega)=[\omega-\mathbf{k}^2+\mu-\mathcal{U}(\mathbf{k}\omega)]^{-1}$, $\mathcal{U}(\mathbf{k}\omega)\theta(\omega)$ tells how the motion of a particle above the Fermi sea is effected and thus bears a relation to the optical model potential. In the usual correspondence (see the preceding section) one defines a dispersion relation by

$$\omega(\mathbf{k})=\mathbf{k}^2-\mu+\mathrm{Re}\mathcal{U}(\mathbf{k}\omega)\theta(\omega), \quad (54)$$

which is assumed to be unperturbed by the imaginary part present in $\mathcal{U}$. The potential describing the scattering of the particle is then

$$V(\mathbf{k})=V_1(\mathbf{k}_1)+iV_2(\mathbf{k}_2),$$

with

$$V_1(\mathbf{k})=\mathrm{Re}\mathcal{U}(\mathbf{k}\,\omega(\mathbf{k})),\qquad V_2(\mathbf{k})=\mathrm{Im}\mathcal{U}(\mathbf{k}\,\omega(\mathbf{k})). \quad (55)$$

Also here one pays for the simplicity of the model. Since T has a pole at $\frac{1}{2}\mathbf{K}^2-2\mu-\epsilon_B$ and a branchpoint at $\frac{1}{2}\mathbf{K}^2-2\mu$, $V_2\equiv 0$ for unoccupied bound states with $0\leqslant\omega\leqslant-\mu$ and for scattering states with $-\mu\leqslant\omega\leqslant-2\mu-\epsilon_B$.

The optical potential is not $V(\mathbf{k})$, (55), but that potential averaged over many resonances. Such a procedure is conceptually even necessary in view of the analytical properties of the exact $G_1(\mathbf{k}\omega)$. From (32), for instance, $G_1(\mathbf{k}\omega)$ is seen to have singularities on the real axis only, whereas a complex $\mathcal{U}(\mathbf{k}\omega)$, which results in general from an approximation, would make the singularities of $G_1(\mathbf{k}\omega)$ [see Eq. (35)] complex. A reconciliation is then achieved by an averaging procedure resulting in a replacement of $\omega(\mathbf{k})$ by $\omega(\mathbf{k})+\frac{1}{2}iV_2(\mathbf{k})$, V_2 being a finite width.[20]

We recall here Bell's remark that proper treatment of the Pauli principle leads to an optical-model wave function expressed in terms of $G_1(\mathbf{k}\omega)$.[21] Whereas Bell had a diagrammatical representation in mind, the clear physical picture given by him is equally valid for a

[20] G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959).

[21] J. S. Bell and E. J. Squires, Phys. Rev. Letters **3**, 96 (1959); J. S. Bell, in *Lectures on Fields Theory and the Many-Body Problem, Naples, 1960* [Academic Press Inc., New York (to be published)].

nonperturbative treatment. The estimate (55) above with the mentioned theoretical shortcoming should be meaningful for a situation where the incident nucleon is not discriminated from the target particles with regard to dynamical correlations as well as the treatment of statistics. We wish further to stress that it belongs to the potentialities of the approach of Koltun and Wilets to derive both the momentum distribution and the optical potential for an infinite medium under constraint, i.e., possessing a density gradient. The variation with density of the above-mentioned quantities and, in particular, the behavior of the optical potential at the nuclear surface, undoubtedly constitute desirable information.

6. COMPARISON AND DISCUSSION

The physical quantities we set out to derive, the polarizability and the response function of a fermion system in its ground state have received ample attention in the past, in particular with regard to the electron gas. Most discussions of nuclear matter for that purpose followed the same systematics, often lacking the necessary rigour.

It seems that the Schwinger-Martin-Puff theory constitutes an approach with fresh content, which without free parameters at its disposal accounts in a satisfactory way for binding energy and equilibrium density. The correct outcome of those quantities is no guarantee for a corresponding agreement for other ground-state properties even if some of them, like the momentum distribution $\rho(\mathbf{k})$ and the pair correlation function $P(\mathbf{k})$, contain, in fact, the information necessary to calculate the ground-state energy. It is, however, tempting to investigate quantities like the system's polarizability, collective states, and optical model within the same framework, since the mentioned quantities appear all related to each other through the simple connection (20) and (21), of G_2 and G_1.

We now turn to a comparison with preceding treatments and in particular the one by Gottfried and Pičman,[18] who also formulate the problem of the linear response in terms of ground-state Green's functions. The cornerstone there is the Migdal-Galitskii integral equation for G_2,[22] which is cited here for comparison

$$G_2(12;1'2')=G_1(11')G_1(22')-G_1(12')G(21') +G_1(13)G_1(2'4)\langle 34|\Gamma|56\rangle G_2(52;1'6). \quad (56)$$

In order to solve for G_2 one is bound to approximate the formal kernel Γ in (56). Gottfried and Pičman suggested an expansion in a series of T matrices. In the low-density limit for short-range forces only one T matrix is retained, which is eventually approximated by a scattering length. This lowest order approximation then amounts to the random-phase approximation for a system of zero range of appropriate strength.[23]

Equation (56) has to be contrasted with the approximate MSP Eq. (24) for G_2, containing one T matrix only, which incidently may not be approximated, since the forces are supposed to be strong enough to bind the system.

The main difference in both approaches, however, lies in the entirely different situations described by them.[24] G_2, Eq. (56), describes the propagation of a particle-hole pair and a collective state (zero sound) may appear. The G_2, solution of Eq. (20), on the other hand, describes the scattering of a pair of particles. Anomalous poles there may correspond to Cooper states.[11,25]

We now turn to the calculation of the polarization energy by Koltun and Wilets.[9] In following Puff consistently, these authors calculate the polarization energy as the shift in the energy of the system including the external field, from which in the end the field energy is subtracted. The two methods should agree if no approximations are made. Since, however, both expansions retain terms proportional only to a_0^2, one might expect differences. The treatment of the density propagator above is, indeed, in the spirit of the original Puff approximation. However, the interpretation of $D(\mathbf{q}0)$ as the polarization energy goes beyond that approximation and assumes the validity of a perturbation treatment of a ground state determined itself in an approximation, namely the MSP scheme. Koltun and Wilets on the other hand calculate, among other things, the modification of the Hartree field and of the ground state. The two resulting expressions are, indeed, dissimilar: In reference 9 the polarization energy appears expressed in terms of a solution of an integral equation, which contains quantities like A, T, $\mathfrak{V}$ for negative frequencies only. Our result is an explicit integral in terms of the same quantities, but of their positive as well as negative frequency portions. Where direct comparison seems to be hard, it will certainly be of interest to see differences and their relative importance in the results of actual calculations, which will be undertaken.

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[22] V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. 34, 139 (1959) [translation: Soviet Phys.—JETP 7, 96 (1958)].

[23] A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. (N. Y.) 6, 1 (1959).

[24] The author wishes to acknowledge an illuminating discussion with Dr. J. S. Bell on this point.

[25] A. S. Reiner, L. Wilets, and D. Koltun (to be published).

Properties of the One-Particle Green's Function for Nonuniform Many-Fermion Systems*

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In this paper we discuss some general properties of the one-particle Green's function, for nonuniform many-fermion systems and the associated single-particle interpretation of physical properties of such systems. We consider, in particular, the ground-state energy, the density of particles in the ground state, and the single-particle excitation spectrum. The investigation is restricted to the case of a static external field and a system at zero temperature. Various general approximation methods are studied starting from one in which the self-energy operator is replaced by an Hermitian and energy-independent operator.

1. INTRODUCTION

BY now, the theory of the Green's function approach to the analysis of uniform many-particle systems has been highly developed through the work of many authors.[1–6] In contrast, the extension of the theory to nonuniform systems, even in the simplest case of a static external potential, has only recently received attention[7–11] and has not yet been treated in a comprehensive manner.

It is true that the generalization of results to the nonuniform case usually turns out to be straightforward. One can fairly say that most of the hard questions—those connected with the specifically many-particle aspects of the problem—have already been answered by treating the uniform case.

Nevertheless, for the purposes of practical calculations on nonuniform systems and their physical interpretation, it is convenient to start from the more general formalism rather than modify the special case. Moreover, certain new qualitative features do arise for nonuniform systems, for example the occurrence of bound single-particle excitations in the presence of an attractive potential even though the interparticle forces are repulsive. In this case, one is concerned with the behavior of the one-particle Green's function alone.

It was, in fact, an example of just this kind—in connection with a perturbative calculation for a dense infinite electron gas in the presence of a point-positive charge[12]—that first drew the author's attention to the problems arising in the Green's function approach to nonuniform systems and to the incomplete nature of the existing treatment of the subject.

For the above reasons, we have thought it worthwhile to emphasize here some general features of the analysis of nonuniform many-fermion systems in terms of properties of the one-particle Green's function, for the case of a static external potential. In the following paper, some of these results and concepts are applied to the above-mentioned problem of the single-particle excitation spectrum of a dense electron gas with a positive point charge.

Let us recall that the one-particle Green's function is defined by

$$G(x,x')=G(x,t;x't')=-i\langle T\{\psi(x)\psi^\dagger(x')\}\rangle. \quad (1)$$

Here, $\psi(x)$ and $\psi^\dagger(x')$ are second quantized Heisenberg operators at the space-time points x,x'; T is the time-ordering operator and the brackets denote an expectation value with respect to the ground state of the many-body system.[13,14] $\psi(x)$ obeys the equation of motion[15]

$$-i\frac{d}{dt}\psi(x)=[H,\psi(x)], \quad (2)$$

where H is the Hamiltonian of the many-particle system.

In the case we are considering, H has the form

$$H=\int d^3x\,\psi^\dagger(x)[-\frac{\nabla_x^2}{2m}+V(x)]\psi(x)$$
$$+\tfrac{1}{2}\int\int d^3x d^3x'\,\psi^\dagger(x)\psi^\dagger(x')v(x-x')\psi(x')\psi(x), \quad (3)$$

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[1] V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 139 (1958); V. M. Galitskii, *ibid.* **34**, 151 (1958) [translations: Soviet Phys.—JETP **7**, 96 and 104 (1958)].

[2] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

[3] A. Klein and R. Prange, Phys. Rev. **112**, 994 (1958).

[4] D. V. Dubois, Ann. Phys. (New York) **7**, 174 (1958); **8**, 24 (1959).

[5] V. Bonch-Bruevich and S. Kogan, Ann. Phys. (New York) **9**, 125 (1960).

[6] T. Kato, T. Kobayashi and M. Namiki, Suppl. Progr. Theoret. Phys. (Kyoto) No. 15, 3–60 (1960).

[7] The nonuniform case is discussed briefly in the fundamental papers of P. Martin and J. Schwinger (reference 2) and T. Kato, T. Kobayashi, and M. Namiki (reference 6). Some of the results of the present paper are closely related to those obtained in the article of Kato, Kobayashi, and Namiki.

[8] J. S. Langer, Phys. Rev. **124**, 997 (1961).

[9] J. S. Langer, Phys. Rev. **124**, 1003 (1961); **120**, 714 (1960).

[10] A. Abrikosov, J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 569 (1961); [translation: Soviet Phys.—JETP **14**, 408 (1962)].

[11] G. Baraff and S. Borowitz, Phys. Rev. **121**, 1704 (1961); G. Baraff, *ibid.* **123**, 2087 (1961).

[12] A. J. Layzer, Bull. Am. Phys. Soc. **6**, 447 (1961).

[13] For systems with degenerate ground states, we define the bracket symbol as including an additional average over the various degenerate ground states.

[14] Spin indices are suppressed. For spin-independent forces, G is diagonal in the spin coordinates and the diagonal elements are equal.

[15] We take $\hbar=1$.